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On a class of the first kind Volterra equations in a problem of identification of a linear nonstationary dynamic system

Svetlana V. SOLODUSHA
Melentiev Energy Systems Institute
of the Siberian Branch of the Russian Academy of Sciences
130 Lermontov St., Irkutsk 664033, Russian Federation

Abstract. This paper proposes an approach to the identification of a nonstationary linear dynamic system. Its input-output mathematical model is presented as a Volterra equation of the first kind. The problem of nonparametric identification of Volterra kernels is solved on the basis of an active experiment using test piecewise linear signals (that have a rising front). The problem statement is based on the conditions for modeling the dynamics of technical devices in the energy and power industry. The choice of an admissible family of input signals is driven by the complexity of generating piecewise-constant type signals for real energy objects. The original problem is reduced to solving Volterra integral equations of the first kind with two variable integration limits. A formula for the inversion of the integral equations under study is constructed. Sufficient conditions are obtained for the solvability of the corresponding equations with respect to Volterra kernels in the class of continuous functions.

Keywords: Volterra equations of the first kind with two variable limits of integration, identification, dynamic system

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НАУЧНАЯ СТАТЬЯ

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О некотором классе уравнений Вольтерра I рода в задаче идентификации линейной нестационарной динамической системы

Светлана Витальевна СОЛОДУША
ФГБУН «Институт систем энергетики им. Л. А. Мелентьева»
Сибирского отделения Российской академии наук
664033, Российская Федерация, г. Иркутск, ул. Лермонтова, 130

Аннотация. В статье предложен подход к идентификации нестационарной линейной динамической системы. Ее математическая модель типа «вход-выход» представлена в виде уравнения Вольтерра I рода. Задача непараметрической идентификации ядер Вольтерра решается на основе активного эксперимента с помощью тестовых сигналов кусочно-линейного вида (имеющих фронт нарастания). Постановка задачи исходит из условий моделирования динамики технических устройств тепло- и электроэнергетики. Выбор допустимого семейства входных сигналов обусловлен сложностью формирования сигналов кусочно-постоянного типа для реальных энергетических объектов. Исходная задача сводится к решению интегральных уравнений Вольтерра I рода с двумя переменными пределами интегрирования. Построена формула обращения выделенных интегральных уравнений. Получены достаточные условия разрешимости соответствующих уравнений относительно ядер Вольтерра в классе непрерывных функций.

Ключевые слова: уравнения Вольтерра I рода с двумя переменными пределами интегрирования, идентификация, динамическая система

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Introduction

The traditional approach of the theory of mathematical modeling involves the use of differential equations based on information about the physical nature of a dynamic object. At the same time, in addition to the input and output variables, the models contain system state variables. At the same time, as noted in [1, p. 11], «detection of differential equations of the ... process is not the only form» of mathematical models. Generally speaking, state variables can be excluded in the transition to models that describe the direct dependence of the output $y(t)$ on the input $x(t)$ [2, p. 27]. A fairly common model of the input–output type in the absence of a priori information about the physical structure of a dynamic system is the Volterra integral equation of the first kind [3]

$$\int_0^t \bar{K}(s)x(t-s)ds = y(t), \quad t \in [0,T], \tag{0.1}$$

where t is time, $x(t)$ is input, $y(t)$ is output, $\bar{K}(s)$ is the Volterra kernel. To build an integral model of the form (0.1) means to solve the problem of restoring the Volterra kernels based on the known data set $x(t)$, $y(t)$. In the theory of automatic control, there is a widespread approach to the identification of $\bar{K}(t)$, based on the use of the Heaviside function $e(t)$. Function $\bar{K}(t) = y'(t)$ (assuming $y'(t) \in C_{[0,T]}$, $y(0) = 0$) is a solution to the equation (0.1) for $x(t) = e(t)$.

A technique based on the applicating the Heaviside functions is developed in the monograph [4] for the case of nonstationary nonlinear dynamic systems, when the transient characteristics (Volterra kernels) change with time t. Following [4], the problem of the identificating $K(t,s)$, $0 \le s \le t \le T$, in the linear equation

$$\int_0^t K(t,s)x(s)ds = y(t), \quad t \in [0,T], \tag{0.2}$$

is reduced to the solution of the integral equation with a variable upper limit of integration ω with respect to the function $K(t,s)$ using piecewise constant test signals of the form

$$x_\omega(t) = e(t) - e(t-\omega), \quad 0 \le \omega \le t \le T. \tag{0.3}$$

As shown in [5,6], the using the left-hand side of (0.2) as a linear term in the Volterra polynomial (segment of an integro-power series) [7] makes it possible to improve the accuracy of modeling the response of stationary nonlinear dynamic systems. Applications should take into account the specifics of technical objects of heat and electric power. As a rule, the signals implemented at the input of the system have a rising edge. In particular, in relation to the objects of heat power [8, p. 8] the input action is conditionally considered to be abrupt if the signal increase in duration does not exceed 10% of the time interval T under study. Thus, the development of methods for constructing the (0.2) model, based on the use of test signal families adapted for applied objects, is of undoubted relevance for solving the problem of identifying transient characteristics for both linear and nonlinear dynamic systems. Note the using the Volterra equations of the first kind of the form (0.2) for nonstationary dynamical systems is limited by the complexity of the problem of identifying the integrand [9].

The purpose of this work is to propose a new method for identifying $K(t,s)$ in (0.2) based on test signals from the class of piecewise linear functions. The paper contents can be conditionally divided into two parts: firstly, the section with the problem statement, secondly, the theoretical one. In the first part, we distinguish new Volterra equations of the first kind with two variable limits of integration that arise in the problem of nonparametric identification of a linear nonstationary dynamical system. These equations allow for explicit inversion formulas. In the second part, sufficient conditions are given that ensure the existence of $K(t,s)$ in the class C_Δ, $\Delta = \{(t,s) : 0 \le s \le t \le T\}$. The conclusion contains the main results of the work.

1. Statement of the nonparametric identification problem

Let the input-output model (0.2) describe the dynamics of a linear nonstationary system with a scalar input $x(t)$ and a scalar output $y(t)$, such that $y(0) = 0$. Assume that the function $y(t)$ in (0.2) is smooth enough to perform the necessary calculations. Developing the technique developed in [4], consider a method for recovering $K(t,s)$ in (0.2) using the one-parameter family

$$\xi_v(s) = \begin{cases} 0, & s \le 0, \\ \dfrac{s}{v}, & 0 < s \le v, \\ 1, & v < s, \end{cases} \tag{1.1}$$

where the parameter $v > 0$ corresponds to the rise time of the front of the test signal [10]. The using piecewise linear input signals is based on taking into account the specifics of setting input actions for technical (energy) objects. The applicability of input signals of this kind in the case when the duration of the rise time of the signal v is constant was considered earlier in [11] when solving the identification problem for $\bar{K}$ from (0.1). In this paper, we consider the situation when the parameter v is variable. This makes it possible to obtain the two-dimensional continuum of initial data required for recovering the unknown values of the function $K(t,s)$.

Assuming that the functions $\xi_v(s)$ belong to the feasible family of test signals of the dynamic system under study, we will perform the substitution $x(s) = \xi_v(s)$ into (0.2). Then the original identification problem $K(t,s)$ is reduced to solving the Volterra equation of the first kind with two variable limits of integration

$$\int_0^v K(t,s)\frac{s}{v}ds + \int_v^t K(t,s)ds = f(t,v), \quad 0 \le v \le t \le T, \tag{1.2}$$

where $f(t,v)$ is the response of the dynamical system to the input of the form (1.1). Note that using a signal in the form (1.1), which is different from (0.3), we complicate the identification problem, since now $x(0) = 0$ and only the derivative $x'(t)|_{t=0} \ne 0$, undergoing a discontinuity, allows us to obtain a well-posed problem on the pair $\left(C, C^{(2)}\right)$. To understand the specifics of (1.2), we formulate the following lemma.

Lemma 1.1. *Let in equation (1.2) with respect to a continuous function $K(t,v)$, nonsymmetric on $\Delta = \{(t,v) : 0 \le v \le t \le T\}$, the right-hand side $f(t,v) \in C_\Delta^{(1)}$ and*

$$f(0,0) = 0. \tag{1.3}$$

Under the condition that $f(t,v)$ exists such that

$$f(t,v) + v\,(f(t,v))'_v = p(t,v), \tag{1.4}$$

holds, equation (1.2) *is equivalent to equation*

$$\int_v^t K(t,s)ds = p(t,v). \tag{1.5}$$

P r o o f. Assuming the validity of the conditions of the lemma, we will show that equivalent transformations allow us to pass from the equation (1.2) to the equation (1.5). Differentiate both parts of (1.2) with respect to v (the operation of differentiation in (1.2) is legal due to condition (1.3)):

$$(f(t,v))'_v = -\int_0^v K(t,s)\frac{s}{v^2}ds.$$

Summing up (1.2) with $v\,(f(t,v))'_v$, we get

$$\int_0^v K(t,s)\frac{s}{v}ds + \int_v^t K(t,s)ds - v\int_0^v K(t,s)\frac{s}{v^2}ds = f(t,v) + v\,(f(t,v))'_v \equiv p(t,v),$$

whence, after reducing similar terms, we have the equation (1.5) with the right-hand side $p(t,v)$ of the form (1.4). □

R e m a r k 1.1. Under the conditions of the Lemma 1.1, the function $f(t,v)$ in the equation (1.2) obtained from (1.5) automatically satisfies the condition (1.3). This follows from

$$\int_0^v K(t,s)\frac{s}{v}ds \to 0 \quad \text{for} \quad v \to 0.$$

In the next section, we consider the problem of identifying the Volterra kernel $K(t,v)$ in (1.2) in more detail.

2. Solvability of a linear Volterra integral equation of the first kind of the form (1.2)

Assuming that (1.2) is uniquely solvable in the class of functions continuous on $\Delta = \{(t,v) : 0 \le v \le t \le T\}$, the solution $K(t,v)$ can be found explicitly [10]:

$$K(t,v) = -\left(2\frac{\partial f(t,v)}{\partial v} + v\frac{\partial^2 f(t,v)}{\partial v^2}\right). \tag{2.1}$$

Consider the question of the existence of a unique continuous solution to (1.2).

Theorem 2.1. *The conditions* (1.3) *and*

$$2\,(f(t,v))'_v + v\,(f(t,v))''_{v^2} \in C_\Delta, \quad \Delta = \{(t,v) : 0 \le v \le t \le T\}, \tag{2.2}$$

are necessary and sufficient for the existence of a solution to equation (1.2), $(t,v) \in \Delta$, *in the class of continuous functions on* Δ. *The solution to* (1.2) *is unique in the specified class and is determined by the formula* (2.1).

P r o o f. *Existence.* Necessity. Let the solution to (1.2) in the class of functions continuous on Δ exist (we denote it by $K^*(t,v)$). It means that

$$f(t,v)=\int_0^v K^*(t,s)\frac{s}{v}ds+\int_v^t K^*(t,s)ds.$$

The fulfillment of the (1.3) condition is obvious. Taking into account (1.4), (1.5) we have

$$f(t,v)+v\left(f(t,v)\right)'_v=\int_v^t K^*(t,s)ds,$$

which ensures that the condition (2.2) is satisfied due to the continuity of K^* on Δ. Thus, the necessity of (1.3), (2.2) is proved.

Adequacy. If (2.2) is true, then the function (2.1) is continuous on Δ. Using a direct substitution of (2.1) into (1.2), we verify that (2.1) turns (1.2) into an identity under the conditions of the theorem. Let us introduce

$$I_1=\int_0^v \frac{\partial^2 f(t,s)}{\partial s^2}s^2ds=v^2\left.\frac{\partial f(t,s)}{\partial s}\right|_{s=v}-2\int_0^v\frac{\partial f(t,s)}{\partial s}sds,$$

$$I_2=\int_v^t \frac{\partial^2 f(t,s)}{\partial s^2}sds=t\left.\frac{\partial f(t,s)}{\partial s}\right|_{s=t}-v\left.\frac{\partial f(t,s)}{\partial s}\right|_{s=v}-f(t,t)+f(t,v).$$

Then substituting $K(t,s)=-\left(2\frac{\partial f(t,s)}{\partial s}+s\frac{\partial^2 f(t,s)}{\partial s^2}\right)$ into (1.2) gives

$$-\int_0^v\left(2\frac{\partial f(t,s)}{\partial s}+s\frac{\partial^2 f(t,s)}{\partial s^2}\right)\frac{s}{v}ds-\int_v^t\left(2\frac{\partial f(t,s)}{\partial s}+s\frac{\partial^2 f(t,s)}{\partial s^2}\right)ds$$
$$=-\frac{2}{v}\int_0^v\frac{\partial f(t,s)}{\partial s}sds-\frac{1}{v}I_1-2f(t,t)+2f(t,v)-I_2. \quad (2.3)$$

By virtue of (1.2),

$$f(t,t)=\frac{1}{t}\int_0^t K(t,s)sds,\qquad \left.\frac{\partial f(t,v)}{\partial v}\right|_{v=t}=-\frac{1}{t^2}\int_0^t K(t,s)sds,$$

therefore, given the condition (1.3),

$$f(t,t)+t\left.\frac{\partial f(t,s)}{\partial s}\right|_{s=t}=0$$

for all $t\in[0,T]$. Therefore, the right-hand side of (2.3) becomes $f(t,v)$.

Uniqueness. Let us prove uniqueness by contradiction. Assume that there are two solutions $K^*(t,v)$ and $K^{**}(t,v)\neq K^*(t,v)$ belonging to C_Δ. Then their difference $\varepsilon(t,v)$ satisfies the identity

$$\int_0^v \varepsilon(t,s)\frac{s}{v}ds+\int_v^t\varepsilon(t,s)ds\equiv 0.$$

For $v = 0$, from this equality we obtain

$$\int_0^t \varepsilon(t,s)ds \equiv 0,$$

therefore, $\varepsilon(t,s) = 0$, so $K^*(t,v) = K^{**}(t,v)$. □

E x a m p l e 2.1. Following [4], it is natural to consider the response of the test dynamic system to the input signal (1.1) in the form

$$y(t,\nu) = \frac{1}{6}\nu^2 M - \frac{1}{2}\nu Mt - \frac{1}{2}\nu + \frac{1}{2}Mt^2 + t, \quad M = \text{const} > 0.$$

It is easy to see that all conditions of the theorem are satisfied, so by (2.1)

$$K(t,s) = 1 + M(t-s)$$

is a solution to (1.2).

Conclusions

The paper considered a new type of Volterra integral equations of the first kind with variable lower and upper limits of integration. Conditions were formulated for the equivalence of distinguished linear integral equations of the form (1.2), associated with the identification of Volterra kernels using piecewise linear input signals, to another type of integral equations of the form (1.5) arising when using signals from the class of piecewise costant functions. A theorem on the existence of a unique solution $K(t,v)$ to the equation (1.2) in the class of continuous on $\Delta = \{(t,v) : 0 \le v \le t \le T\}$ functions was formulated and proved. Further development of the work is connected with the application of the proposed approach in the problem of identifying the transient characteristics of nonlinear dynamic objects of thermal power engineering.

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Information about the author

Svetlana V. Solodusha, Doctor of Engineering, Associate Professor, Head of Laboratory, Melentiev Energy Systems Institute, Irkutsk, Russian Federation. E-mail: solodusha@isem.irk.ru
ORCID: https://orcid.org/0000-0001-6162-7542

Информация об авторе

Солодуша Светлана Витальевна, доктор технических наук, доцент, заведующий лабораторией, Институт систем энергетики им. Л. А. Мелентьева СО РАН, г. Иркутск, Российская Федерация. E-mail: solodusha@isem.irk.ru
ORCID: https://orcid.org/0000-0001-6162-7542

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